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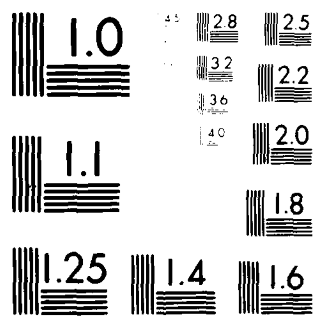

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AFWAL-TR-80-3046

THE TREATMENT OF THE SONIC LINE AND OF SHOCKS IN A FINITE ELEMENT APPROACH TO TRANSONIC FLOW COMPUTATIONS

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Even if one uses the same shape functions throughout in a finite element computation of transonic flow fields, one must, for reasons of stability, apply the weights differently in the subsonic and supersonic regions. In the present report this aspect is accepted without further discussion. It is necessary, in addition, to impose special conditions at the sonic line and at the line where one returns from supersonic to subsonic speeds. These

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20. ABSTRACT (continued)

conditions are first explored by means of a semidiscretization. (The differential operators are discretized with respect to the direction normal to the streamlines, but not with respect to the streamline direction.) The resulting system of ordinary differential equations has singular points, whose position is related to the transition from one regime to the other. These singularities are compatible with weak solutions of the problems. (Finite difference and finite element solutions are realizations of the concept of weak equality.) But from the physical point of view these singularities are not admissible, at least not in the transition from a subsonic to a supersonic speed. One thus obtains the requirement that at the sonic line the partial differential equation be satisfied in the strong sense. This gives additional conditions, which are first expressed in the semidiscretized version of the problem and later introduced directly into the fully discretized finite element version. The condition so obtained is the analogue to Murman's sonic operator. The singularities mentioned above cannot be avoided in the return from supersonic to subsonic speeds. Here the requirement of a weak solution is appropriate. Admitting such singularities one introduces an arbitrariness into the problem which is necessary to satisfy the boundary conditions at the subsonic downstream boundary of the flow field. The admission of suitable weak solutions is equivalent to shock conditions. In the present context it is required that the weights be constant across the shocks. Some adjustment of the weight areas is needed in order to match the number of unknowns with the number of equations. The shock conditions so obtained are correct only in the average over several elements. This might lead to a loss of accuracy unless one chooses the elements sufficiently small. A method is sketched which removes this restriction by taking the shock position within the elements into account. A formulation of this kind will most probably be applied in combination with a Newton-Raphson procedure, in which one determines simultaneously by direct elimination corrections for the entire flow field including the position of the shock from a system of linear equations.

FOREWORD

The work was performed in 1979 under Grant AFOSR-78-3524 to the University of Dayton for the Applied Mathematics Group, Analysis and Optimization Branch, Structures and Dynamics Division, Flight Dynamics Laboratories under Project 2304N1 and Work Unit 2304N110. Dr. Karl G. Guderley was Principal Investigator.

The author expresses his appreciation for the efficient typing work of Miss Norma Harting and for the art work of the staff of the University of Dayton.

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INTRODUCTION

The computation of transonic flow fields by means of finite difference methods has reached a high degree of sophistication. One of the most successful approaches (that of Jameson, Ref. 1) alternates between a Poisson solver and an iteration process patterned after that of Murman and Cole (Ref. 2, 3). The Poisson solver can hardly be improved by the finite element concept, but there is a chance that the task performed by the Murman Cole iteration can be carried out more efficiently. The present report does not claim to have achieved this goal. It asks a preliminary question: which special measures are needed in order to obtain such a finite element formulation. The approach taken is primarily theoretical. The procedures which emerge are more complicated than those of Murman and Cole. Whether simplifications are possible remains to be seen. The author considers it preferable first to study what should be done in principle and to make simplifications and compromises only after the basic procedure has been established, although a less systematic approach may sometimes be equally successful.

There is no doubt that a finite element approach is applicable in subsonic problems, for the computation can be based on an extremum principle. Such a secure basis does not exist in the supersonic region, but the study of examples shows that stable methods using a finite element concept exist (Ref. 4). Two approaches are possible: one can either treat the subsonic and the supersonic regions separately and then impose matching conditions, or one can try to develop a procedure in which the subsonic and the supersonic regions are treated simultaneously, although not exactly in the same manner. Even in such an approach the transition from the subsonic to the supersonic region and back requires special measures. In finite difference form this is the approach of Murman and Cole. In the present report the analogous finite element version will be studied. Of particular importance are the conditions that must be imposed at the shock and at the sonic line. The sonic operator and the shock operator of the Murman-Cole procedure can be interpreted as the finite difference realization of the ideas that will be developed.

SECTION II

CONDITION AT THE SONIC LINE OBTAINED BY A SEMI-DISCRETIZATION

In Reference 5 the author has shown that a special condition must be introduced at the sonic line. We ask here how this condition expresses itself in a finite element setting. Reference 5 is motivated by the observation that the boundary conditions for the flow in a channel are quite different in a purely supersonic and in a purely subsonic flow. In a supersonic flow one prescribes the full velocity vector in the entrance cross section. In a subsonic flow one has conditions in the entrance and in the exit cross section. If one has a transition from a subsonic to supersonic flow, one has subsonic boundary conditions in the entrance and no boundary conditions in the exit cross section. In some manner, one must have extra conditions within the flow field which are substitutes for the conditions which would be given in the entrance cross section if the flow were entirely supersonic. The nature of these conditions is recognized if one treats the problem by means of a semidiscretization; the differential operator for the direction normal to the streamlines is discretized, but the derivatives in the streamwise direction are retained. One obtains a system of ordinary differential equations. To be specific, let us assume that the flow field is described by the values of the potential at a discrete set of streamlines. (Mostly we think of the simplified form of the potential equation then the streamlines are lines y = const. For the full potential equation, one will introduce a linearization and the streamlines are those of the preceding iteration step.) One then obtains a system of ordinary differential equations for the potential along these selected streamlines. One finds that singular points arise where these lines intersect the sonic line.

At these singular points all but one of the solutions of the homogeneous system are regular. The singularity of this exceptional solution is compatible with a weak solution of the system of equations, but it is not admissible because then the streamwise component of the velocity gradient becomes infinite. The missing condition for the system of differential equations is obtained by the requirement that such behavior of the solution is excluded. In mathematical terms one would say, that at the sonic line the differential equation must be satisfied in the strong sense. For the nonlinearized differential equation this amounts to the requirement that the second derivative normal to the streamline of the potential be zero. This is the condition imposed by Murman. For the linearized equation which one must solve when one applies a Newton Raphson procedure, the condition is slightly more complicated. It is best derived directly from the linearized differential equation. The following analysis differs somewhat from that of Reference 5 because it refers to a partial differential equation which resembles more closely the one which arises by a linearization of the original non-linear problem.

We shall deal here with either bilinear or bicubic shape functions in a rectangular grid system. For functions of this character one can carry out the discretization for the direction normal to the streamlines and for the streamline direction separately. Then one obtains as an intermediate form the system of ordinary differential equations mentioned above. In practice this intermediate step will be omitted, but it is worthwhile to discuss it because it suggests the form of the final numerical formulation. The description of the practical procedure which will be given in Section III is selfcontained. A reader primarily interested in this aspect may pass over the present discussions, whose main purpose is to provide a motivation.

We start from the simplified equation for transonic flow

$$-(\gamma+1)\Phi_x\Phi_{xx} + \Phi_{yy} = 0$$

Assume that one has an approximation

$$\Phi(x,y) = F(x,y)$$

and that one tries to obtain corrections by means of a Newton Raphson procedure. Then one sets

$$\Phi(x,y) = F(x,y) + \phi(x,y)$$

and obtains by a linearization

$$-(\gamma+1)\ [F_x\phi_{xx} + F_{xx}\phi_x] + \phi_{yy} -(\gamma+1)F_xF_{xx} + F_{yy} = 0$$

Set

$$G(x,y) = -(\gamma+1)F_x$$

and

$$h(x,y) = -(\gamma+1)\ F_xF_{xx} + F_{yy}$$

Then one has

$$G(x,y)\phi_{xx} + G_x(x,y)\phi_x + \phi_{yy} + h(x,y) = 0 \qquad (1)$$

In Reference 5 the term containing ϕ_x has been omitted. This changes the details of the argument, but not the overall conclusions. Now a semidiscretization is carried out with respect to the y direction, either by a finite element procedure or by a finite difference approximation and one arrives at a system

$$A(x)\psi'' + A'(x)\psi' + B\psi + f(x) = 0 \qquad (2)$$

Primes denote differentiation with respect to the independent variable x; ψ and f are x dependent vectors. The components of ψ are the parameters which serve to describe the potential along lines x = const. (If one chooses, for instance, to discretize by means of a finite difference procedure, then the components of $\psi(x)$ are given by the values of $\phi(x,y)$ for the pivotal lines y = const chosen for the discretization. In this case the components of f are the values of h(x,y) along those lines.) If the potential along a line x = const is described by n parameters, then ψ and f have n components. A and B are x dependent matrices. (In the case of a finite difference approximation A is a diagonal matrix with diagonal element $G(x,y_n)$. B is a tri-diagonal matrix (with elements $1/h^2$, $-2/h^2$ and $1/h^2$), it arises from the finite difference form of ϕ_{yy}.

Incidentally, Eq. (1) can be obtained from a variational formulation, namely

$$\delta\iint (G(x,y)\phi_x^{\ 2} + \tfrac{1}{2}\phi_y^{\ 2} - h(x,y)\phi)\ dxdy = 0$$

(with suitable boundary conditions). The differential equation from which one starts in Ref. 5 does not possess this property. The system Eq. (2) is now written as a first order system.

$$\bar{\bar{A}}\psi' + \bar{\bar{B}}\psi + \bar{f} = 0 \tag{3}$$

where

$$\bar{\psi} = \begin{bmatrix} \psi \\ \hline \psi' \end{bmatrix}, \quad \bar{f} = \begin{bmatrix} 0 \\ \hline f \end{bmatrix}, \quad \bar{A} = \left[\begin{array}{c|c} I_n & 0 \\ \hline 0 & A \end{array}\right], \quad \bar{B} = \left[\begin{array}{c|c} 0 & -I_n \\ \hline B & A' \end{array}\right] \tag{4}$$

The system Eq. (3) can obviously be solved for $\bar{\psi}'$ if the determinant $|\bar{A}| = |A|$ is different from zero. Singular points are encountered if it vanishes. To study this case consider the eigenvalue problem

$$\bar{A}\ \bar{u} - \lambda\bar{u} = 0 \tag{5}$$

or in more detail with $\bar{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$$u_1 - \lambda u_1 = 0$$

$$Au_2 - \lambda u_2 = 0$$

It has n trivial eigensolutions

$$\lambda = +1, \; u_2 \equiv 0, \; u_1 \text{ arbitrary}$$

Another set of eigensolutions has

$$u_1 \equiv 0$$

while λ and u_2 satisfy

$$Au_2 - \lambda u_2 = 0$$

The further development is best carried out on the basis of the original eigenvalue problem, Eq. (5). We characterize the eigenvectors and eigenfunctions by subscripts and consider in addition the adjoint equation

$$\bar{A}\bar{u}_n - \lambda_n \bar{u}_n = 0 \tag{6}$$

$$\bar{v}_n^T \bar{A} - \lambda_n \bar{v}_n^T = 0 \tag{7}$$

In this notation, the vectors are regarded as n by 1 matrices. All quantities occurring in these equations depend upon the independent variable x. The determinant of the matrix $\bar{A}$ vanishes at those values of x where one of the eigenvalues vanishes. Let the values of x where the eigenvalue $\lambda_k(x)$ vanishes be denoted by x^k. One has (for all values of x) the orthonormality conditions

$$\bar{v}_\ell^T \bar{u}_k = \delta_{\ell k} \tag{8}$$

Then it follows that

$$\bar{v}_\ell^T A\bar{u}_k = \lambda_k \delta_{\ell k} \tag{9}$$

We represent $\bar{\Psi}'$ in the form

$$\bar{\psi}' = \sum_k c_k \bar{u}_\kappa \tag{10}$$

Then one finds, using Eqs. (8) and (9)

$$c_k = -\frac{\bar{v}_k^T (\bar{B}\bar{\psi} + \bar{f})}{\lambda_k} \tag{11}$$

At a point $x = x^m$, where λ_m vanishes, one obtains a finite value of $\bar{\psi}'$ only if

$$\bar{v}_m^T (\bar{B}\bar{\psi}+\bar{f}) = 0 \tag{12}$$

The value of $\bar{\psi}'(x^m)$ is determined provided that Eq. (12) which restrict the choice of $\bar{\psi}$ is satisfied, although Eq. (11) gives an undetermined expression for $c_m(x^m)$. Whether or not this value of $\bar{\psi}'$ is needed depends upon the method used for solving the systems Eq. (2) or Eq. (3). In the usual predictor corrector methods one evaluates the derivatives at the chosen grid points, then $\bar{\psi}'$ must be evaluated, in a finite element approach the derivatives do not appear explicitly, and one can forgo the computation.

The derivative $\bar{\psi}'$ at a point $x = x^m$ can be computed in the following manner. We define the auxiliary quantity

$$\tilde{\psi}' = \sum_{k \neq m} - \frac{\bar{v}_k^T (\bar{B}\bar{\psi}+\bar{f})\,\bar{u}_k}{\lambda_k}$$

Then from Eq. (10)

$$\bar{\psi}' = \tilde{\psi}' + c_m \bar{u}_m \tag{14}$$

One has, because of Eq. (8)

$$v_m^T \tilde{\psi}' = 0$$

For $x = x^m$ one can compute $\tilde{\psi}'$ by solving the equation

$$(\bar{A}+\alpha u_m \bar{v}_m^T)\tilde{\psi}' = -(\bar{B}\bar{\psi}+\bar{f}) \tag{15}$$

$$\tilde{\psi}' = -(\bar{A}+\alpha \bar{u}_m \bar{v}_m^T)^{-1} (\bar{B}\bar{\psi}+\bar{f})$$

provided that Eq. (12) is satisfied. That the expression Eq. (15) is correct is seen because the matrix $(\bar{A} + \alpha\bar{u}_m\bar{v}_m^{+})$ has the same eigenvectors as $\bar{A}$ and also the same eigenvalues except for k = m; there one has the eigenvalue $\lambda_m + \alpha$. The right hand side of Eq. (15) is therefore given by

$$\tilde{\psi}' = \sum_{k\neq m} \frac{\bar{v}_k^T (\bar{B}\bar{\psi}+\bar{f})}{\lambda_k} \bar{u}_k - \frac{v_m^T (\bar{B}\bar{\psi}+\bar{f})}{\lambda_m+\alpha} \bar{u}_m$$

The second term vanishes because of Eq. (12). Now we must determine c_m in Eq. (14). One obtains, by applying L'Hospital's rule to Eq. (11)

$$c_m = - \frac{(d\bar{v}_m^T /dx)(\bar{B}\bar{\psi}+\bar{f}) + \bar{v}_m^T ((d\bar{B}/dx)\bar{\psi} + (d\bar{f}/dx)) + \bar{v}_m^T \bar{B}(d\bar{\psi}/dx)}{d\lambda_m/dx} \tag{16}$$

To compute the quantities occurring in this expression we differentiate Eq. (7):

$$\frac{d\bar{v}_m^T}{dx} \bar{A} - \frac{dv_m^T}{dx} \lambda_m + \bar{v}_m^T \frac{d\bar{A}}{dx} - \frac{d\lambda_m}{dx} \bar{v}_m^T = 0$$

The solvability condition for this inhomogeneous system is

$$\bar{v}_m^T \left(\frac{d\bar{A}}{dx} - \frac{d\lambda m}{dx}\right) \bar{u}_m = 0$$

Hence

$$\frac{d\lambda_m}{dx} = \bar{v}_m^T \frac{d\bar{A}}{dx} \bar{u}_m \tag{17}$$

Furthermore, since $\lambda_m(x^m) = 0$

$$\frac{dv_m^T}{dx} \bar{A} = - \bar{v}_m^T \left(\frac{d\bar{A}}{dx} - \frac{d\lambda_m}{dx}\right)$$

Hence, employing an argument similar to the one used above

$$\frac{dv_m^T}{dx} = - \bar{v}_m^T \left(\frac{d\bar{A}}{dx} - \frac{d\lambda_m}{dx}\right) (A+\alpha u_m v_m^T)^{-1} + \tilde{c}\bar{v}_m^T \tag{18}$$

Here $\tilde{c}$ is arbitrary (ultimately determined by a normalization condition). The constant $\tilde{c}$ vanishes upon substitution of Eq. (18) into Eq. (16), because of Eq. (12). Multiplying Eq. (16) by $d\lambda_m/dx$ and substituting Eq. (14) one finds

$$c_m\,[\bar{v}_m^T\,\frac{dA}{dx}\,\bar{u}_m] - \bar{v}_m^T\,(\frac{d\bar{A}}{dx} - (\bar{v}_m^T\,\frac{dA}{dx}\,\bar{u}_m))(\bar{A} + \alpha\bar{u}_m\bar{v}_m^T\,)^{-1}(\bar{B}\bar{\psi}+\bar{f})$$

$$+\bar{v}_m^T\,(\frac{d\bar{B}}{dx}\psi + \frac{d\bar{f}}{dx}) - \bar{v}_m^T\,\bar{B}(\bar{A}+\alpha\bar{u}_m\bar{v}_m^T\,)^{-1}(\bar{B}\bar{\psi}+\bar{f}) + c_m\bar{v}_m^T\,B\bar{u}_m = 0$$

Hence

$$c_m = [\bar{v}_m^T\,(\frac{dA}{dx} + B)\bar{u}_m]^{-1}\,\{\bar{v}_m^T\,(\frac{d\bar{A}}{dx} - (\bar{v}_m^T\,\frac{d\bar{A}}{dx}\,\bar{u}_m) + \bar{B})(A+\alpha\bar{u}_m\bar{v}_m^T\,)^{-1}(\bar{B}\bar{\psi}+\bar{f})$$
$$-\bar{v}_m^T\,(\frac{d\bar{B}}{dx}\bar{\psi} + \frac{d\bar{f}}{dx})\} \tag{18}$$

This formula expresses c_m in terms of $\bar{\psi}$ and the known quantities $\bar{A}$, $\bar{B}$, $d\bar{A}/dx$, $d\bar{B}/dx$, $\bar{f}$ and $d\bar{f}/dx$. As was mentioned above, $\bar{\psi}'$ is not always needed.

To impose the condition that the solution of the system is free of singular points, one must first determine the values of x^k for which $\lambda_k = 0$. For these points one must evaluate $\bar{u}_k$ and $\bar{v}_k$ and, for methods in which $\bar{\psi}'$ is needed, a number of derivatives of known quantities.

For an illustration, consider the second order equation

$$-x\phi'' - \phi' - \phi + f = 0$$

With

$$\psi_1 = \phi,\ \psi_2 = \phi^1 \tag{19}$$

one obtains the equivalent first order system

$$\begin{bmatrix} 1 & 0 \\ 0 & -x \end{bmatrix}\begin{bmatrix} \psi_1' \\ \psi_2' \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ f \end{bmatrix} = 0 \tag{20}$$

The associated eigenvalue problem, Eq. (5), is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & -x \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \lambda \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

The eigenvalues and eigenfunctions are found by inspection

$$\lambda_1 = 1; \quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad ; \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$\lambda_2 = -x; \quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad ; \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

A singular point occurs at $x = 0$, ($\lambda_2(x)$ is zero at this point). The regularity condition Eq. (12) gives

$$[0 \; \vdots \; 1] \begin{bmatrix} -\psi_2 \\ -\psi_1 - \dot{\psi}_2 + f \end{bmatrix} = 0$$

or

$$-\psi_1(0) - \dot{\psi}_2(0) + f(0) = 0$$

$$-\phi(0) - \phi'(0) + f(0) = 0 \tag{21}$$

This result can be found directly from Eq. (19). The derivative $\psi_2'(0)$ computed from Eq. (20),

$$-x \, \psi_2' - \psi_2 - \psi_1 + f = 0$$

is undetermined. The desired result is obtained by a differentiation with respect to x

$$\psi_2'(0) = (1/2)(-\psi_1'(0) + f'(0)) \tag{22}$$

moreover

$$\psi_1'(0) = \psi_2(0)$$

The essence of a numerical procedure which computes the solution in an interval $-a < x < 0$, with $\phi(-a)$ prescribed is already seen if one traverses this interval in one step. One has in the simplest difference formulation of Eq. (19)

$$\frac{a}{2}\,\frac{\phi(-a)-2\phi(-a/2)+\phi(0)}{(a/2)^2} - \frac{\phi(0)-\phi(-a)}{a} - \phi(-a/2) + f(-a/2) = 0 \qquad (23)$$

The value of $\phi(-a)$ is given by the boundary conditions. In addition one has the regularity condition for $x = 0$. It is consistent, if one expresses the value of $\phi'(0)$ by a finite difference approximation.

$$\phi'(0) = \frac{3\phi(0)-4\phi(-a/2)+\phi(-a)}{a} \qquad (24)$$

The values of $\phi(-a/2)$, $\phi(0)$, and $\phi'(0)$ can then be computed from Eqs. (23), (24) and (21).

In the first step of the continuation to positive values of x a formula corresponding to Eq. (24) is used

$$\phi'(0) = \frac{-3\phi(0)+4\phi(a/2)-\phi(a)}{a}$$

The left side is given by Eq. (24). In addition, one has from the difference approximation of the differential equation for the point $x = a/2$

$$\frac{a}{2}\,\frac{\phi(0)-2\phi(a/2)+\phi(a)}{(a/2)^2} - \frac{\phi(a)-\phi(0)}{a} - \phi(a/2)+f(a/2) = 0$$

The last two equations are used to compute $\phi(a/2)$ and $\phi(a)$. From there on the values of ϕ can be computed in sequence.

The procedure is even simpler if the grid points straddle the point $x = 0$. Assume that $\phi(-3/4a)$ is prescribed. Then one has, for the first inner point

$$\frac{a}{4}\,\frac{\phi(3a/4)-2\phi(-a/4)+\phi(a/4)}{(a/2)^2} - \frac{\phi(a/4)-\phi(-3a/4)}{a} - \phi(-a/4)+f(-a/4) = 0$$

In the regularity condition, derived from Eq. (19), ϕ_0 is expressed either by linear interpolation

$$\phi(0) = (1/2)[\phi(-a/4)+\phi(+a/4)]$$

or by quadratic interpolation

$$\phi(0) = -(1/8)\phi(-3a/4)+(3/8)\phi(a/4)+(3/4)\phi(-a/4)$$

Moreover

$$\phi'(0) = \frac{\phi(a/4)-\phi(-a/4)}{a/2}$$

One thus has two equation which allow one to compute $\phi(-a/4)$ and $\phi(+a/4)$. For a continuation to further positive values of x, one uses a marching procedure. The value of $\phi(3a/4)$ is, for instance, obtained in terms of $\phi(-a/4)$ and $\phi(+a/4)$, from the difference equation for the point $x = a/4$.

The situation is similar if one treats Eq. (20) instead of Eq. (19) by a difference formulation. Taking as grid points $x = -a$, $x = -a/2$ and $x = 0$, one obtains as equations for the middle of the intervals $-a < x < -a/2$ and $-a/2 < x < 0$

$$\frac{\psi_1(-a/2)-\psi_1(-a)}{a/2} - \frac{\psi_2(-a/2)+\psi_2(-a)}{2} = 0$$

$$\frac{\psi_1(0)-\psi_1(-a/2)}{a/2} - \frac{\psi_2(0)+\psi_2(-a/2)}{2} = 0$$

$$\frac{3a}{4}\,\frac{\psi_2(-a/2)-\psi_2(-a)}{a/2} - \frac{\psi_1(-a/2)+\psi_1(-a)}{2} - \frac{\psi_2(-a/2)+\psi_2(-a)}{2} + f(-\tfrac{3a}{4}) = 0$$

$$\frac{a}{4}\,\frac{\psi_2(0)-\psi_2(-a/2)}{a/2} - \frac{\psi_1(0)+\psi_1(a/2)}{2} - \frac{\psi_2(0)+\psi_2(-a/2)}{2} + f(-\tfrac{a}{4}) = 0$$

The value of $\psi_1(-a)$ is given as boundary condition. In addition, one has the regularity condition (first of Eqs. (21)). One thus has 6 equations for 6 unknowns (ψ_1 and ψ_2 at each of the points $x = -a$, $x = -a/2$ and $x = 0$). For the continuation to the right one uses a marching procedure which considers one interval at a time.

No essential differences are encountered, if one obtains discretized equations by a finite element procedure. If one takes, for instance, third degree shape functions, then one characterizes the solution by the parameters $\phi(-a)$, $\psi'(-a)$, $\phi(0)$, $\phi'(0)$, etc. For each interval one has two weight functions (because one has two parameters for each grid point). In addition, $\phi(-a)$ is assigned and one has the regularity condition (second of Eqs. (21)). This suffices to compute the solution for $x < 0$.

Eq. (22) for the derivative at $x = 0$ would be needed if one uses formulae analogous to those occurring in the usual integration techniques for ordinary differential equations. Using a very simply integration procedure and traversing the interval $-a < x < 0$ in two steps one has

$$\psi_2(-a/2)-\psi_2(-a) = (a/2)(1/2)(\psi_2'(-a/2)+\psi_2'(-a))$$

$$\psi_1(-a/2)-\psi_1(-a) = (a/2)(1/2)(\psi_1'(-a/2)+\psi_1'(-a))$$

$$\psi_2(0) - \psi_2(-a/2) - (a/2)(1/2)(\psi_2'(0)+\psi_2'(-a/2))$$

$$\psi_1(0) - \psi_1(-a/2) = (a/2)(1/2)(\psi_1'(0)+\psi_1'(-a/2))$$

At the points $-a$ and $-a/2$, ψ_1' and ψ_2' are expressed in terms of ψ_1 and ψ_2 by means of the differential equations (20). At $x = 0$, these quantities are expressed by Eqs. (22). They are derived from Eq. (20). Thus, one has four relations for six quantities (ψ_1 and ψ_2 at three points), $\psi_1(-a)$ is given by the boundary condition. In addition, one has the regularity condition (21).

Next, we consider an example which arises from a partial differential equation. Consider

$$(y-a(x))\phi_{xx}-a'(x)\phi_x+\phi_{yy} = 0 \tag{25}$$

For $y > a(x)$ the problem is elliptic, for $y < a(x)$ it is hyperbolic. The line $y = a(x)$ will be called the parabolic line. We choose as boundary condition, $\phi = 0$ for $y = 0$, and $y = 1$. Let $a(x)$ be a function which changes in the region under consideration monotonically from some value less than zero to a value exceeding 1. Then one has values of x at the left where the problem is elliptic and at the right where it is hyperbolic. In the intermediate region one finds a parabolic line which goes from the lower left to the upper right. Let

$$y_n = \frac{n}{N}$$

$(n < N)$. We approximate $\phi(x,y)$ by means of the values which it assumes along the "pivotal" lines, $y = y_n$.

$$\phi_n(x) = \phi(x,y_n)$$

Taking a finite difference approximation for ϕ_{yy}, one then obtains the system of ordinary differential equations

$$(y_n-a(x))\phi_n'' - a'(x)\phi_n' + N^{-2}(\phi_{n+1}(x) - 2\phi_n(x) + \phi_{n-1}(x)) = 0 \tag{26}$$

$$n = 1, 2 \ldots N - 1$$

The matrix A introduced in Eq. (1) is a diagonal matrix with elements $y_n - a(x)$, and the augmented matrix $\bar{A}$ is also a diagonal matrix with additional elements 1. Eigenvalues 0 occur, obviously, at values of x, to be denoted by x^n, where

$$a(x^n) = y_n$$

These are the intersection of the pivotal lines with the parabolic line. It is obvious from the form of the differential equations, that these are the singular points. Let

$$\vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} \leftarrow \text{nth component}$$

be the unit vector in the direction of the n^{th} component of the space of the vectors ϕ. One has as eigenvectors of $\bar{A}$, aside from those with eigenvalue 1 which are not of interest

$$\bar{u} = \begin{bmatrix} 0 \\ -- \\ \vec{e}_m \end{bmatrix} , \quad \bar{v}_m = \begin{bmatrix} 0 \\ -- \\ \vec{e}_m \end{bmatrix}$$

with eigenvalues $\lambda_m(x) = y_m - a(x)$. In this case one has for the matrix B

$$B = N^{-2} \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \vdots & \vdots & & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix}$$

The regularity condition (12) at a singular point $x = x^m$ gives

$$(0 \mid \vec{e}_m) \bar{B} \begin{bmatrix} \psi_1 \\ -- \\ \psi_2 \end{bmatrix} = (0 \mid \vec{e}_m) \begin{bmatrix} -\psi_2 \\ \\ -a'(x_m)\psi_2' + B\psi_1 \end{bmatrix} = -a'$$

Hence

$$-a'(x^m)\psi_{2,m} + N^{-2}[\psi_{1,m-1}(x^m) - 2\psi_{1,m}(x^m) + \psi_{1,m+1}(x^m)] = 0 \tag{27}$$

This condition can also be found by inspection of Eq. (26). One needs 2n boundary conditions for the system of ordinary differential equations Eq. (26). n conditions are given by the values of $\phi(x_\ell, y_n)$, where x_ℓ is the value of x at the left end of the region under consideration, n further conditions are obtained from the regularity conditions Eq. (27).

The form of the regularity conditions is more complicated if one applies a different form of discretization, although it is based on the same idea. One might choose a finite element approximation for the y direction, or (even though this may not be practical) a representation in terms of a finite Fourier analysis based on the values of ϕ_n (see Ref. 5) In those cases, the matrix A does not have diagonal form, and the determination of the eigenvalues and eigenvectors v_n is more difficult. In Ref. 5 it is shown by an example that for large values of N (that is if the pivotal lines y = const lie close together) the functions of y represented by the eigenvectors of A approach δ functions. Thus, one obtains in essence the same eigenfunctions as in a finite difference discretization. They are used again to derive regularity conditions for the points $x = x^n$.

The system of ordinary differential equations so obtained is now treated by a discretization with respect to x by the same techniques described above for a simpler case.

The weight functions applied in the x discretization combined with the weight functions for the y discretization give two dimensional weight functions. Figures 1 and 2 show the boundaries of the regions to which the weights are applied. For constant weights the boundaries of adjacent areas coincide. If one works with nonconstant weights then there is in general an overlap of the weight areas, but the general arrangement is the same.

Figure 1 refers to bilinear shape functions which are characterized by the values of ϕ at the corners of the quadrangular elements. To each of the inner lines y = const which form an element boundary there belongs one differential equation and for each of these lines one needs one regularity condition. In the supersonic region it is probably necessary for reasons of stability to shift the weight areas downstream. This has not been indicated in the figure. Each row of

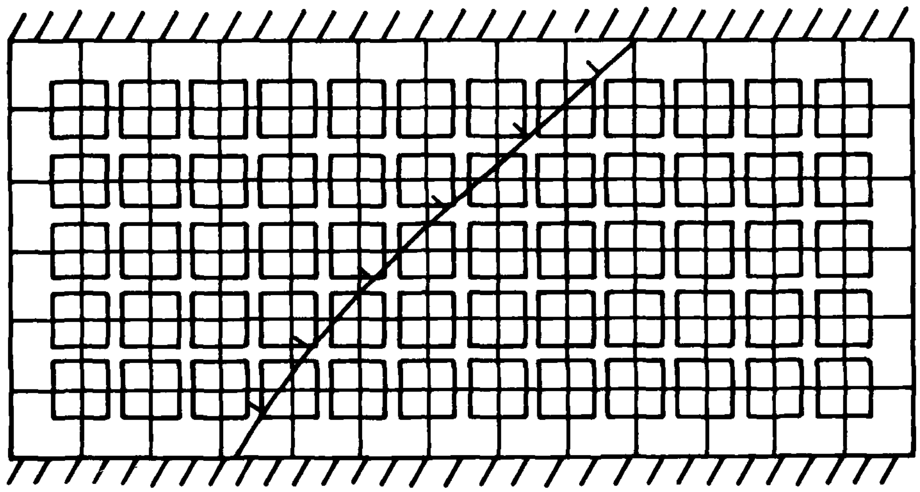

Figure 1. Weight Areas for Bi-Linear Shape Functions, Element Boundaries and Sonic Line.

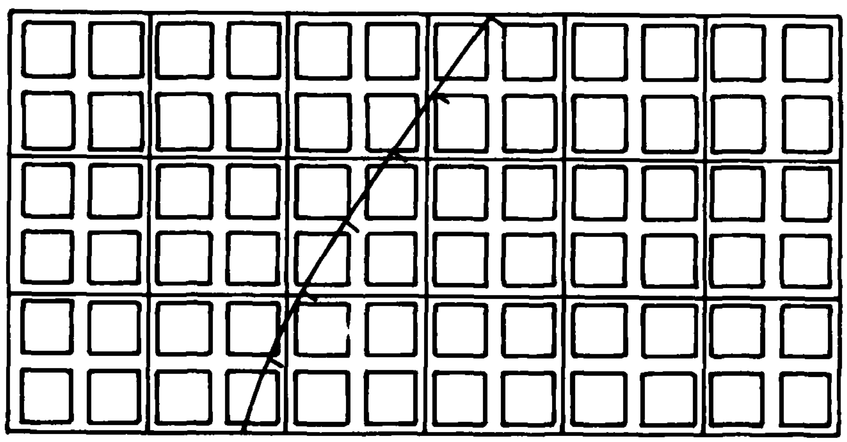

Figure 2. Weight Areas for Bi-Cubic Shape Functions, Element Boundaries and Sonic Line.

weight areas correspond to one differential equation. (The differential equations will, of course, contain terms which originate from adjacent lines y = const.) For each of the unknowns, that is for each row of weight areas one has one regularity condition. For Eq. (1) it is the requirement that $G_x\phi_x + \phi_{yy} + h(x,y) = 0$ at the parabolic line.

For bicubic elements the same situation is shown in Fig. 2. The lengths of the sides of the quadrangles is taken twice those of the bilinear elements. Bicubic elements can be characterized by the values of ϕ, ϕ_x, ϕ_y, and ϕ_{xy} at each element corner. For each inner corner one therefore has four weight areas. At the upper and lower boundaries the values of ϕ are known, in addition one knows at the left boundary the values of ϕ_y and at the upper and lower boundaries the values of ϕ_x. One has therefore two weight areas for each element corner at the boundary of the region. Again, each row of weight areas corresponds to one differential equation. And for each such row one obtains from the system of ordinary differential equation one regularity condition. In the form derived above these conditions are inconvenient. They require that one determine the values of x^n (those values of x where the eigenvalues $\lambda(x)$ vanish), and the eigenfunction v_n of the adjoint operator $\bar{A}$. The exact form of these conditions depends upon the chosen discretization procedure. These conditions express in discretized form the fact that (for the present example) $-G_x\phi_x + \phi_{yy} + h(x,y) = 0$ at the parabolic line. The procedure derived above is one possible form which it can be expressed. It is simpler to introduce this requirement directly. Details of such a procedure are shown in the next section.

If one wants to insure overall conservation of mass, then one must use constants as weight functions and cover the entire region of integration with weight areas, neither gaps nor overlap are permitted. Usually, conservation of mass is needed only for shocks.

In Ref. 5 the question has been raised of what happens if lines x = const intersect certain characteristics more than once. For a more complicated partial differential equation this can easily happen in the vicinity of the parabolic line, where the two families of characteristics have a common tangent. If one uses in such a case the semidiscretization technique described above, then one obtains, again, a system of differential equations with singular points, but these singular points lie in the supersonic region. The typical phenomena associated with the parabolic line are postponed by some kind of a sweep effect introduced by the orientation of the coordinate system. This approach is questionable because one can find examples in which it seems to give solutions to boundary value problems which are not well posed.

The question whether it is necessary to choose the element boundaries so that this phenomenon cannot occur, or whether it suffices if one chooses the boundaries of the weight areas so that the regions of dependence are properly taken into account in the weighting of the residuals is left open.

SECTION III

DIRECT TREATMENT OF THE CONDITIONS FOR THE SONIC LINE

The discussion in Section II started from the differential equation (1) obtained by linearization of the transonic version of the potential equation

$$G(x,y)\phi_{xx} + G_x(x,y)\phi_x + \phi_{yy} + h(h,y) = 0$$

A discretization is carried out by introducing a rectangular mesh, expressing ϕ by either bilinear or bicubic shape functions and by applying weights (say constant weights) according to the patterns shown in Figures 1 and 2, respectively for bilinear and bicubic shape functions. Along the parabolic line, that is along the line where $G(x,y) = 0$, the differential equation yields the relation

$$G_x\phi_x + \phi_{yy} + h(x,y) = 0 \tag{28}$$

Eq. (28) will be used to derive further equations in addition to those obtained by weighting of the residual. One can assume that the weight areas shown in Figures 1 or 2 arise in two steps by carrying out a discretization in the y-direction first and in the x-direction afterwards. In the first step one obtains a system of ordinary differential equations, one for each row of weight areas. For each differential equation and therefore also for each row of weight areas one needs one additional condition. These are obtained from Eq. (28). These conditions are written in the form

$$\int(G_x\phi_x + \phi_{yy} + h(x,y))ds = 0 \tag{29}$$

where the integrals are formed over the segments of the parabolic line which extend over the width of the rows of weight areas. This provides as many conditions as there are differential equations. The discussions of Section II have the purpose of showing that these conditions arise naturally from the system of differential equations obtained by the y-discretization. Proceeding in this manner, one obtains the system symbolized for bilinear shape functions by Fig. 3. The shape functions are determined by the values of ϕ at the corners of the quadrangles. These values have been denoted by $\phi_{j,k}$ where j is the subscript for the element boundaries x = const, and k for the element boundaries y = const. The values of $\phi_{j,k}$ for constant j (that is the values of ϕ along the j^{th} line x = const) are combined into a vector ψ_j. Part of the matrix shown in Fig. 3 is block tridiagonal, the blocks are square with dimensions determined by the dimension of the vector ψ_j. The first row contains only two blocks because at the entrance cross section the vector ψ is given by the boundary conditions, its contribution to the system of equations appears in the inhomogeneous part. The last row of the square blocks contains three because the vector ψ in the exit cross section is unknown. The last row consists of single equations. They arise from the regularity conditions Eq. (28). The segment of the sonic line which lies within one row of weight areas may intersect more than one weight area. Accordingly, the conditions expressed by Eq. (28) will give relations between a number of vectors ψ_j, but usually only a very small number of components of these vectors will appear.

The sparsity of this matrix can be taken into account fairly easily. In principle, one eliminates the vectors ψ_j in the sequence given by their subscripts. The elimination includes, of course, the equations obtained from the regularity conditions Eq. (28). In this process there comes a points where some equations obtained from the regularity conditions contain after the eliminations only components of the first of the remaining vectors ψ_j. It is then possible

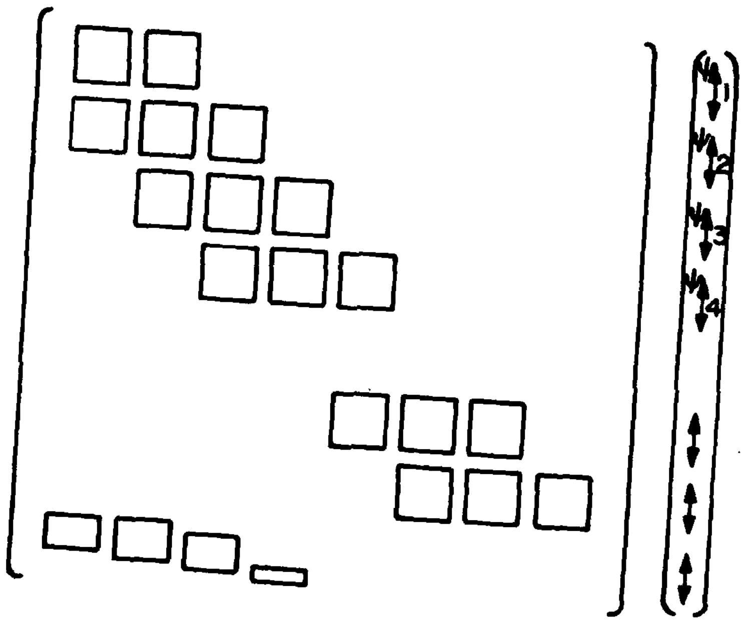

Figure 3. Schematic Form of the Matrix Which Arises by Direct Application of the Conditions for the Sonic Line.

to express some of the components of this vector in terms of the other components. While in the initial elimination process one has to solve, in each step, systems of equations whose size is determined by the dimension of ψ_j, one performs from this point on this task in two steps; in the first one, some of the components of the vector ψ_j are expressed in terms of the others, in the second one, one eliminates the remaining components from the system. After all regularity conditions have been taken into account in this manner, one arrives at a system in which one can compute the remaining vectors ψ_j directly in sequence. This means that one carries out a marching procedure.

SECTION IV

TRANSITION FROM A SUPERSONIC TO A SUBSONIC FLOW

For a first orientation we consider, again, the ordinary differential equation (19)

$$-x\phi'' - \phi' - \phi = 0$$

or rather

$$x\phi'' + \phi' - \phi = 0 \tag{30}$$

and with a slight generalization

$$a(x)\phi'' + a'(x)\phi' - \phi = 0, \quad a(0) = 0, \quad a'(x)>0 \text{ for } x>0 \tag{30a}$$

because it is natural to make the transition from the oscillatory type of the particular solutions (which is analogous to a supersonic flow) to the nearly monotonic type (analogous to subsonic flows) as one proceeds in the positive x direction from a negative to a positive value. The left end of the region in which the solution is to be found is taken at a negative value of x. There one prescribes the values of ϕ and ϕ', in analogy to boundary conditions for supersonic problems. The right end of the region is taken at some positive value of x, and in accordance with boundary conditions for elliptic problems one prescribes only the value of ϕ (or ϕ'). While we had too few boundary conditions in the discussions of Section II (1 instead of 2), we have now too many (3 instead of 2). The differential equation has a singular point at $x = 0$. We found in Section II, that for the transition from the "elliptic" to the "hyperbolic" type the missing condition is provided by postulating regularity of the solution at $x = 0$. Under the present condition regularity cannot be achieved because for negative values of x the solution is completely determined by the initial conditions at the left end of the interval. We ask whether one can satisfy the boundary conditions at the right end, if at $x = 0$ one satisfies the differential equations only in the weak sense.

In Eq. (30a) the coefficient $a(x)$ is regular and has a zero of the first order at $x = 0$. Particular solutions of Eq. (30a) can be written in the following manner

$$\phi_1 = P_1(x) \tag{31}$$

and

$$\phi_2 = \log|x|P_1(x) + cxP_2(x) \tag{32}$$

where $P_1(x)$ and $P_2(x)$ are power series in x which have 1 as constant term, and c is a suitable constant. Incidentally, Eq. (30) is solved by

$$\phi = Z_0(ix^{\frac{1}{2}}) \qquad x > 0$$
$$\phi = Z_0(|x|^{\frac{1}{2}}) \qquad x < 0$$

where Z_0 is a linear combination of the Bessel functions J_0 and N_0. The general solution is

$$\phi(x) = c_1^+\phi_1(x) + c_2^+\phi_2(x) \quad , \qquad x > 0 \tag{33}$$

$$\phi(x) = c_1^-\phi_1(x) + c_2^-\phi_2(x) \quad , \qquad x < 0$$

c_1^+, c_2^+, c_1^-, c_2^- are constants. In the region $x < 0$ the solution and therefore also c_1^- and c_2^- are determined by the conditions prescribed at the left end of the interval. Eq. (30a) can be written as

$$\frac{d}{dx}\left(a(x)\phi'(x)\right) - \phi = 0$$

Then by integrating from $x = -\varepsilon_1$ to $x = +\varepsilon_2$ ($\varepsilon_1 > 0$, $\varepsilon_2 > 0$)

$$a(\varepsilon_2)\phi'(\varepsilon_2) - a(-\varepsilon_1)\phi'(\varepsilon_1) + \int_{-\varepsilon_1}^{\varepsilon_2} \phi(x)dx = 0 \tag{34}$$

The integral vanishes in the limit $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$, if one expresses ϕ by Eqs. (33), (31) and (32).

To evaluate the first two terms we form

$$\phi'(x) = c_1^{\pm}P_1'(x) + c_2^{+} [\frac{1}{x} P_1 + \log x\ P_1' + cP_2 + cxP_2']$$

Then, since $a(0) = 0$, and since $P_1(0) = 1$

$$\lim_{\varepsilon \to 0} (a(\varepsilon)\phi'(\varepsilon)) = c_2^{\pm}\ a'(0)$$

Therefore, from Eq. (34)

$$c_2^{+} = c_2^{-}$$

while c_1^{+} remains undetermined.

The requirement that at $x = 0$ the differential equation be satisfied only in the weak sense introduces an indeterminacy which makes it possible to satisfy the boundary condition prescribed at the right end of the interval.

The numerical procedure is quite clear if one treats the differential equation by some standard numerical approach for the solution of ordinary differential equations. For negative values of x, one has an initial value problem which is solved up to some negative value of x close to zero. In the vicinity of $x = 0$ one computes two linearly independent particular solutions for ϕ by means of their development and determines the values of c_1^{-} and c_2^{-} by matching at some negative value of x with the solution obtained by numerical integration. For positive values of x, one starts from $x = 0$, setting $c_2^{+} = c_2^{-}$ and leaving c_1^{+} arbitrary; c_1^{+} is determined by the boundary conditions at the right end.

In studying which form the procedure assumes if the differential equations are solved by a finite element approach, let us assume that the elements straddle the point $x = 0$. Choosing the weight regions for $x < 0$ in a manner suitable for hyperbolic problems and for $x > 0$ in a manner suitable for elliptic problems, the number of equations exceeds the number of unknowns by one (Fig. 4a). The weighting of residuals is a finite dimensional approximation to the concept of weak solutions, the present procedure is basically in conformance with the concept developed above, but it gives too many conditions. This is remedied by omitting the weight region which covers the first finite element boundary at a positive value of x and by extending the preceding weight area so that it covers this boundary (Fig. 4b). This treatment of the vicinity of $x = 0$ is the analogue to Murman's shock operator.

The linearized form of the transonic equation Eq. (1) which served as basis for the discussions carried out so far, is complicated because the function G in Eq. (1) is discontinuous at the shock location. For a first orientation we therefore consider the nonlinearized transonic equation

$$-(\gamma+1)\phi_x\phi_{xx} + \phi_{yy} = 0$$

Fig. 5 show the choice the weight areas for such a case. For bilinear elements one has delta function residuals at the element boundaries. The weight areas must overlap the lines at which delta functions occur. In the supersonic region one best uses an infinitesimal overlap at the upstream end of the elements, but not at the downstream end. In the subsonic region, the weight areas are chosen symmetric (in the x and in the y directions) with respect to the element boundaries. In elements which contain the shock, the weight areas overlap the downstream as well as the upstream boundary. The first weight areas of subsonic type start at the downstream boundary of the weight area pertaining to the shock. In the shock weight areas the weight is taken constant to guarantee conservation of mass.

Figure 4a. Weight Areas for an Ordinary Differential Chosen According to the Type (Oscillatory or Nonoscillatory) of Particular Solutions (More Equations than Unknowns).

Figure 4b. Modification of Figure 4a in which one Weight Area is Eliminated.

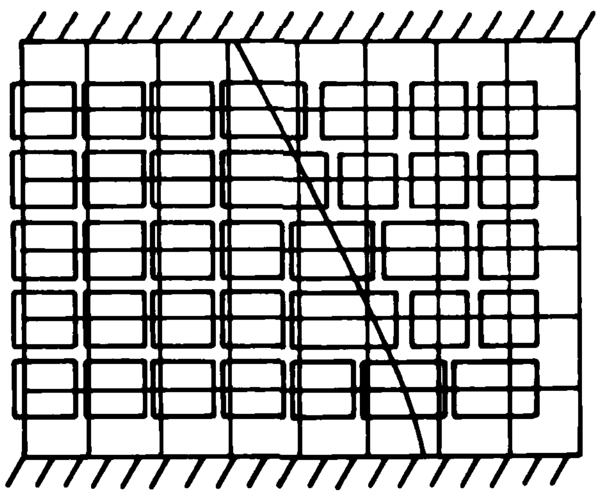

Figure 5. Weight Areas for the Transition From Supersonic to Subsonic Velocities by Means of a Shock.

In examining whether mass is conserved through the shock, one is hampered by the fact that shape functions which are cutomarily applied are not suited to express the sudden change of velocity which occurs in a shock. Conservation of mass can be shown only in the average over several rows of weight areas, and this requires that the states in front and behind the shock change only little along the shock and that the location of the shock relative to the weight areas in the upper and in the lower rows is the same. Then the inadequacies in the representation of the flow field, in particular of ϕ_y, cancel each other and one obtains that, in the average, mass is conserved. This is exactly Murman's argument.

The shock operator is, of course, compatible with a smooth transition from a supersonic to a subsonic flow. It is therefore applicable to shockfree flow fields.

The fact that the shock conditions are satisfied only in the average over several elements makes the choice of rather small elements mandatory. At least in the stage of the computations where a relatively good approximation has already been found and one wants to refine this by using elements in which the shock position is identified. A shock is without influence on the supersonic region upstream. The supersonic region can actually be continued downstream beyond the shock location. (An obvious example is the bow shock in front of a blunt body in a supersonic parallel flow.) In the supersonic flow, the state of points which lie in the area of influence of the data at a preceding station x = const is independent of the boundary conditions which one may prescribe at the upper or lower boundary of the region. This is approximately correct, even if one applies it in an implicit method, as is frequently done in this context. One therefore can find the supersonic region ahead of the shock and to some extent, its continuation downstream with the usual finite element formulation for supersonic flows using fictitious boundary conditions at the upper end (this description refers to a shock on the upper side of a profile). Once the flow in the supersonic region ahead of the shock is known, the complete state of the flow immediately behind the shock depends only upon the position of the shock. For the present discussion, we are using rectangular elements and consider in lieu of ordinary differential equations the difference equations corresponding to rows of weight areas bounded by lines y = const. The shock can be described by means of the x coordinates of its intersection with these row boundaries. These coordinates are now considered as unknown. In the simplest case one can approximate the shock between the intersection points by straight lines. In this manner one obtains one additional free parameter for each row of weight areas. This gives exactly the number of additional unknowns to satisfy the boundary conditions at the downstream end of the subsonic region.

An insight into the character of the problem can be obtained in the following manner: Assume that the shock position is known. Then one has in the region downstream of the shock a purely subsonic problem. A finite element formulation will lead to a system of equations which contains among the unknowns the potential at the shock, while the mass flux through the shock (which is temporarily assumed to be known) appears as a contribution to the inhomogeneous terms. The solution to this problem would be completely determined. Actually, the normal component to the mass flow is unknown because the shock location is not known, but it and also the potential can be expressed in terms of the shock abscissas. The potential is continuous and normal component of the mass flux are continuous through the shock. This gives two sets of additional conditions in terms of the shock abscissas which are considered as additional parameters in the description of the flow field. The procedure is best used in the form of a Newton Raphson iteration for the whole flow field in which one computes simultaneously the potential in the subsonic part of the flow and corrections to the shock location. This idea will be developed in more detail in a report where the possibilities of a separate treatment of the subsonic and the supersonic regions with subsequent matching are discussed. The development sketched last applies only to that part of the shock where the flow behind the shock is subsonic.

The flow field in the vicinity of the junction between the shock and the sonic line is rather complicated. The author considers the description developed by Nocilla as correct (Ref. 6) while he believes that the solutions of Germain (Ref. 7) which are sometimes quoted in this context refer to a different setting, namely the reflection of a sonic line in the form of a shock of a singularity propagating toward the sonic line (see the discussions in Ref. 8). These details are probably not within the capabilities of present flow computation methods. On the other hand, it is questionable whether a detailed computation is worth the effort from a practical point of view.

The determination of the flow field described here is best carried out in the form of a Newton Raphson iteration in which one solves the corrections to an existing approximation by direct elimination. In this context the following observation may be of interest. Using an extremum formulation and bilinear elements, the author has tried to solve a purely subsonic problem by column relaxation similar to that of Murman for finite difference equations. Since one seeks the extremum of a certain functional, convergence is guaranteed, just as it is in Murman's treatment. However, convergence turned out to be disappointingly slow. This can be explained as follows. Consider the example of the Laplace operator. In a finite difference procedure, one obtains the following difference star

$$\Delta x^{-2} \begin{bmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \Delta y^{-2} \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and for $\Delta x = \Delta y$

$$\Delta x^{-2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

In column relaxation one changes only the elements of the middle column which then leads to a tridiagonal matrix

$$\begin{bmatrix} -4 & 1 & & \\ 1 & -4 & 1 & \\ & 1 & -4 & 1 \\ & & \vdots & \end{bmatrix}$$

For bilinear finite elements one obtains the following stars

$$\Delta x^{-2} \begin{bmatrix} 1/6 & -1/3 & 1/6 \\ 2/3 & -4/3 & 2/3 \\ 1/6 & -1/3 & 1/6 \end{bmatrix} + \Delta y^{-2} \begin{bmatrix} 1/6 & 2/3 & 1/6 \\ -1/3 & -4/3 & -1/3 \\ 1/6 & 2/3 & 1/6 \end{bmatrix}$$

and for $\Delta x = \Delta y$

$$\Delta x^{-2} \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & -8/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

For column relaxation one then deals with a matrix

$$(1/3)\begin{bmatrix} -8 & 1 & & \\ 1 & -8 & 1 & \\ & 1 & -8 & 1 \\ & & \vdots & \end{bmatrix}$$

Here one has much stronger diagonal dominance than for finite differences; that is, one is closer to point (over) relaxation. This probably explains the slowness of convergence. The situation is aggraved if one takes $\Delta y > \Delta x$. Naturally, this argument does not apply if one uses direct elimination
A question of convergence still remains but only because the problem is nonlinear.

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